Microwave instability beyond threshold

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Stability of the steady-state bunch distribution described by the Haissinski solution [J. Haissinski, Nuovo Cimento **18B**, 72 (1973)] is studied above the threshold of microwave instability. It is shown that instability may lead to a new self-consistent state corresponding to particles trapped in a separatrix of an unstable mode. The free energies of the two solutions are compared. The relaxation oscillations between the new and Haissinski solutions are possible and may be related to the sawtooth instability observed recently in the experiments [P. Krejcik *et al.*, (unpublished)]. [S1063-651X(96)11109-0]

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I. INTRODUCTION

The microwave instability is one of a few problems of accelerator physics which is not fully understood today. The instability is usually described as an increase of the rms energy spread of a bunch when N_B, the number of particles per bunch, exceeds some threshold value. Because the equilibrium temperature is determined by the damping and the noise of the synchrotron radiation, which is normally independent of N_B, increase of the temperature indicates that there is some additional noise or a mechanism which pumps energy from the longitudinal motion to the uncorrelated single-particle motion.

Recent experiments [2,3] found new features of bunch behavior at the threshold of the instability, such as relaxation oscillations of the rms bunch length (the sawtooth instability of the SLAC damping ring), and large periodic oscillations of the rms size and bunch centroid in LEP. Similar phenomena were also previously observed in different laboratories. Such a behavior is not trivial for a system with damping, which usually goes, after some relaxation time, to an equilibrium steady state. It is reasonable to think that these phenomena are related to the microwave instability and can give insight to its origin. In fact, the new instabilities can be considered as a special case of the microwave instability when only a few azimuthal modes are involved, which may significantly simplify theoretical consideration of the problem. A possible phenomenological explanation of the sawtooth instability may be based on the idea of the ''overshoot phenomena,'' where an unstable mode is stabilized by nonlinear processes at large amplitudes or by bunch heating produced by decaying mode. Radiation damping and filamentation brings the system back to original state. The relaxation oscillations may arise under proper relationship between the growth rate of an unstable mode, its filamentation time, and the synchrotron radiation damping time. When there are many interacting unstable modes, the sawtooth instability becomes a microwave instability. Although generically this is a correct picture, a detailed model of the processes is needed. Our attempts to obtain the sawtooth behavior within the quasilinear approximation in numerical experiments were unsuccessful: the system asymptotically approaches a steady-state with higher temperature.

Study of the microwave instability is usually based on the linearized Vlasov equation. This allows us to find a threshold N_B, spectrum of the eigenmodes, and the rise time of instability. This approach does not, however, describe the dynamics at large amplitudes and is not sufficient to explain the essential nonlinear phenomena, such as the sawtooth instability. Numerical tracking is not very efficient in describing the sawtooth instability, probably because the number of particles involved is a small part of the total bunch population and because simulations with a realistic damping time are computationally prohibitive.

The sawtooth instability indicates the existence of two steady state solutions for large N_B. Depending on the damping time, the system may have relaxation oscillations between these two solutions or may drift adiabatically from one solution to another one provided the damping rate is large enough. Recently, Baartman and Dyachkov [4,5] suggested a model of the sawtooth instability driven by quantum fluctuation in a case of a self-consistent potential having two minima. The mechanism considered in this paper is different; it is related to a nonlinear self-consistent regime arising as a result of a nonlinear resonance. Consideration follows the papers of Shonfeld [6] and Meller [7]. Although this approach does not describe the full time evolution of a bunch, it gives some understanding of the nonlinear dynamics of the unstable modes and, hence, is complimentary to the studies of the linearized Vlasov equation.

II. HAISSINSKI SOLUTION

A single-bunch longitudinal dynamics of particles in a storage ring may be described by canonical variables x,p, where $x=z/\sigma_0$ is the position of a particle in a bunch in units of the zeros current rms bunch length σ_0, and the canonically conjugated momentum $p=-\alpha\delta/\sigma_0$, proportional to $\delta=\Delta E/E$ and the momentum compaction factor α. The coordinate $x=0$ corresponds to the equilibrium rf phase $\psi_{\rm rf}$; $x>0$ for a particle in the head of a bunch. The Hamiltonian in these variables is

$$H(x,p,t)=\frac{p^2}{2}+U(x,t), \quad U(x,t)=U_{\rm rf}+U_W. \qquad (1)$$

The total potential $U(x)$ is the sum of the rf potential $U_{\rm rf}\simeq\omega_{0s}^2x^2/2c_0^2$ and the self-consistent potential U_W:

$$U_W=\lambda\int_x^\infty dx'\rho(x',t)S[(x'-x)\sigma_0]. \quad (2)$$

Here $\omega_{0s}=\omega_{rev}\nu_s$ is the zeroth current synchrotron frequency, the distribution function $\rho(x,t)=\int dp\rho(x,p,t)$ is normalized to 1, $\int dx\rho(x)=1$, and $S(z)$ is an integral of the wake function $W(z)$, $S(z)=\int_0^z dz'W(z')$. $W(z)$ defines energy loss per turn of a particle trailing another particle at the distance z: $\Delta E(z)=-N_b e^2 W(z)$. The wake $W(z)=0$ for $z<0$, and can be expressed in terms of the beam impedance $Z(\omega)$ that has poles in the upper plane of ω. The factor λ depends on the number of particles per bunch N_b,

$$\lambda=\frac{\alpha N_b r_0}{2\pi R\gamma\sigma_0^2}. \quad (3)$$

The distribution function $\rho(x,p,t)$ is a solution of the Fokker-Plank equation

$$\frac{\partial\rho}{\partial t}+\{H,\rho\}=\left[D\frac{\partial^2\rho}{\partial p^2}+\gamma_d\frac{\partial}{\partial p}p\rho\right], \quad (4)$$

which includes effects of diffusion and damping due to synchrotron radiation not described by the Hamiltonian. Derivation of the Fokker-Plank equation and further references can be found in the recent paper [8].

Haissinski solution [1] is the steady state solution of Eq. (4),

$$\rho_H(x,p)=\frac{1}{Z_t}\exp-H(x,p)/T, \quad (5)$$

where $T=D/\gamma_d$. Equation (5) can be factorized $\rho_H(x,p)=\rho(p)\rho(x)$. Hence, $\langle p^2\rangle=T(\sigma_0^2/\alpha)^2$ and is independent on the distortion of the self-consistent potential $U(x)$. The distribution function is

$$\rho_H(x)=\frac{1}{Z_H}e^{-U(x)/T}, \quad \int\rho_H(x)dx=1. \quad (6)$$

At the zero-current, $\rho_H(x)$ describes a Gaussian bunch with the rms $\langle x^2\rangle=Tc_0^2/\omega_{0s}^2$ By the definition of σ_0, $\langle x^2\rangle=1$ at the zero-current, giving

$$T=\omega_{0s}^2/c_0^2, \quad \sigma_0^2\omega_{0s}^2=c_0^2\alpha^2\langle\delta^2\rangle. \quad (7)$$

The explicit form of the solution can be obtained analytically only for a few impedances. In the general case, Eq. (6) can only be solved numerically.

III. LINEARIZED VLASOV EQUATION

The Haissinski solution describes very well the deformation of the bunch shape with N_b (the so-called potential well distortion). Generally speaking, it formally exists for arbitrary N_b. Experiments show that $\langle p^2\rangle$ starts growing when N_b exceeds some threshold value, indicating that above the threshold, the Haissinski solution becomes unstable.

Different authors have given different criteria for the onset of this instability. The threshold N_b is usually defined by the criterion

$$\sqrt{\frac{2}{\pi}}\frac{r_0 N_B}{Z_0\alpha\gamma\delta^2\sigma}\left|\frac{Z}{n}\right|_{\rm eff}\leqslant 1. \quad (8)$$

Here $n=\omega/\omega_{rev}$, $Z_0=120\pi\Omega$, and the ''effective impedance'' $(Z/n)_{\rm eff}$ is related to the machine impedance Z/n either through the experimental ''SPEAR scaling'' or by defining Z/n as weighted with the bunch spectrum. Other criteria have been discussed as well that, i.e., the instability occurs when the slope of the total voltage becomes zero within a bunch length. Bane [9] successfully used this criterion for calculating the threshold of the microwave instability for the SLC damping ring. There have been attempts to relate the threshold of the instability to the appearance of the second minima in the potential well at large bunch currents. Another criterion of the threshold is given by N_b at which the synchrotron frequency as a function of amplitude has extremum, $d\omega(J)/dJ=0$. These criteria give different thresholds of instability. For example, for a self-consistent potential approximated by a polynomial $U(x)=x^2[\omega_0^2/2+\alpha x/3+\beta x^2/4]$ with parameters varying with N_b, ω' changes sign at $\alpha^2/(\beta\omega^2)>0.9$, the second minimum appears at $\alpha^2/(\beta\omega^2)>4.0$, and the potential at this minimum is smaller than $U(x)$ at $\alpha^2/(\beta\omega^2)>4.5$.

Study of the stability usually is based on the linearized Vlasov equation obtained from Eq. (4) for small $f(x,p,t)=\rho(x,p,t)-\rho_H(x,p)$, neglecting the right-hand side (RHS) of the equation. This gives a homogeneous equation which defines azimuthal and radial eigenmodes of perturbation, and gives their frequencies. The onset of the microwave instability is related to a mode-coupling, when some of the eigenfrequencies become complex.

More detailed phenomenology of the microwave instability is based on the quasilinear approach that takes into account the feedback effect of the growing unstable mode on the self-consistent potential, which may stop the instability.

Alternatively, the growing mode leads to a new quasi-steady-state solution. We illustrate the origin of the new solution in the quasilinear approximation. The distribution function is split into two functions $\rho(x,p,t)=\rho_0(x,p,t)+f(x,p,t)$, with slow and fast dependence on time, correspondingly. The Fokker-Plank equation then gives two equations:

$$\frac{\partial\rho_0}{\partial t}+\{H(\rho_0),\rho_0\}=\left[D\frac{\partial^2\rho_0}{\partial p^2}+\gamma_d\frac{\partial(p\rho_0)}{\partial p}\right]-\overline{\{U_W(f),f\}}, \quad (9)$$

$$\frac{\partial f}{\partial t}+\{H(\rho_0),f\}+\{U_W(f),\rho_0\}=0, \quad (10)$$

where $H(f)$ means that the self-consistent Hamiltonian is calculated with the function f, and the bar in the first equation means averaging over the fast oscillations. The second equation can be simplified by a canonical transform from x,p to the angle-action variables ϕ,J such that $H(\rho_0)=H(J)$.

Equation (10) defines the azimuthal harmonics $f_m(J)$,

$$f(J,\phi,t)=\sum_m f_m(J)e^{i\Omega t-im\phi},$$

$$f_m(J)=\frac{\partial\rho_0}{\partial J}\frac{U_m(J)}{\omega(J)-(\Omega/m)}, \tag{11}$$

where $\omega(J)=dH(J)/dJ$, and

$$U_m(J)=\int\frac{d\phi}{2\pi}e^{im\phi-i\Omega t}U_W(f). \tag{12}$$

Substituting Eq. (11) into Eq. (12) gives a dispersion relation defining real and imaginary parts of $\Omega=m(\omega_r+i\gamma_m)$, $m>0$, provided $\omega(J)>0$. A mode is unstable if $\gamma_m<0$. Note that $\gamma_m\propto\partial\rho_0/\partial J$.

Equation (11) describes dependence of a mode on J and, for small increments γ_m, shows that a mode is localized around the resonance value J_r defined by $\omega_r\equiv\omega(J_r)$. The sign of m is the same as the sign of Ω because $\omega(J)>0$.

Equation (9) for the zeroth azimuthal harmonics $\rho_0(J)$ in the new variables takes the form

$$\frac{\partial\rho_0}{\partial t}=\frac{\partial}{\partial J}\left[D\frac{J}{\omega(J)}\frac{\partial\rho_0}{\partial J}+\gamma_d J\rho_0(J)\right]+\frac{\partial}{\partial J}\left[\overline{\frac{\partial U_W(f)}{\partial\phi}f}\right]. \tag{13}$$

The last term in the RHS describes the feedback effect of a mode on the distribution function:

$$\overline{\frac{\partial U_W(f)}{\partial\phi}}f=\frac{2m\gamma_m|U_m(J)|^2}{(\omega(J)-\omega_r)^2+\gamma_m^2}\frac{\partial\rho_0}{\partial J}e^{-2m\gamma_m t}. \tag{14}$$

This term can be combined with the term proportional to diffusion coefficient D; it can, therefore, change the bunch temperature and change the self-consistent potential. As a result, the unstable growing mode may either be stabilized or it may decay.

There is, however, another possibility: the distribution function can come to a new equilibrium when $\gamma_m\to 0$. The modification of the distribution function corresponds to the well-known results of the quasilinear theory in plasma: $\partial\rho_0/\partial J\propto\gamma_m\to 0$ at $J=J_r$. It is closely related to the Van-Kampen waves in the theory of Landau damping. This solution is considered below.

IV. RESONANCE SOLUTION

Here we show that the Fokker-Plank equation has, in addition to the Haissinski solution, another solution, which we call the resonance solution [6]. We use notation ρ_M for this solution following Meller's study [7] of the thermal instability.

Suppose there is an azimuthal mode excited to a finite amplitude, with frequency Ω. Such a mode can be considered as the periodic perturbation for particles in a bunch. Resonance particles, with synchrotron frequencies $\omega(J)\simeq\Omega/n$, if there are any, may be trapped in a separatrix. Motion of the trapped particles produces a periodic modulation of the wake field and of the bunch density. We want to find a self-consistent solution where resonance particles produce a periodic perturbation supporting trapping of these particles in the separatrix. This mechanism describes the nonlinear regime of Landau damping.

Consider a spontaneous small perturbation to the Hamiltonian Eq. (11) that corresponds to excitation of the n-s azimuthal harmonic of the self-consistent potential:

$$H(J,\phi,t)=H(J)+[V_n(J)e^{i(\Omega t-n\phi)}+\text{c.c.}]. \tag{15}$$

Close to a threshold of the instability, each azimuthal mode can be expected to be independent. This corresponds to the assumption that separatrices produced by individual modes do not overlap. Oide's numerical analysis [12] of the instability confirms this assumption.

At the zero amplitude $V_n=0$, particles rotate in the phase plane with frequency $d\phi/dt=\omega(J)=dH/dJ$, and the steady-state distribution is given by the Haissinski solution

$$\rho(J)=\frac{1}{Z_H}e^{-H(J)/T}. \tag{16}$$

Suppose now that there is an amplitude J_r within the bunch length for which $\omega_r\equiv\omega(J_r)=\Omega/n$. In this case, the perturbation V_n in the Hamiltonian produces a resonance and cannot be removed by a canonical transformation. As it is well known, however, the problem can be solved by reducing it to a time-independent problem. This can be done by a canonical transformation to the coordinate q,α,

$$J=J_r+q,\quad \alpha=\phi-\omega_r t+\xi-\arg(V_n)/n. \tag{17}$$

Here, $\xi=\pi/n$ or $\xi=0$, depending on the sign of $\omega_r'\equiv(d\omega/dJ)_{J_r}$, and $V_n=|V_n|e^{i\arg V_n}$. The new Hamiltonian is the Hamiltonian of a pendulum

$$H(q,\alpha)=\frac{q^2}{2M}-\epsilon\cos(n\alpha), \tag{18}$$

where ϵ can be considered as a constant,

$$\epsilon=2|V_n(J_r)|,\quad \frac{1}{M}=|\omega_r'|. \tag{19}$$

It is convenient to introduce a parameter κ defining the energy

$$H=\frac{2\epsilon}{\kappa^2},\quad q=\pm\sqrt{\frac{4M\epsilon}{\kappa^2}[1-\kappa^2\sin^2(n\alpha/2)]}. \tag{20}$$

The motion with $\kappa>1$ is bounded in the range $|\sin(n\alpha/2)|<1/\kappa$, corresponding to motion within a separatrix. Motion with the energy $0<\kappa<1$ is unbounded; it corresponds to particles above the separatrix with $J>J_r$ for the upper sign, and below the separatrix $J<J_r$ for the negative sign in Eq. (20).

The resonance Hamiltonian Eq. (18) can be made independent of phases using a canonical transform to a variables r,ψ with the generating function

$$\Phi(r,\alpha)=\pm\frac{4}{n}\sqrt{\frac{\epsilon M}{\kappa^2}}E\left(\frac{n\alpha}{2},\kappa\right), \tag{21}$$

where E is the elliptic integral; see Appendix A. The Fokker-Plank equation in the variables r,ψ has the approximate form [see Eq. (78) in Appendix B]

$$\frac{\partial\rho}{\partial t}+\{H(r),\rho\}_{r,\psi}=\gamma_d\frac{\partial}{\partial r}\left[T\left\langle\left(\frac{\partial x}{\partial\psi}\right)^2\right\rangle+r\rho\right]. \quad (22)$$

The equation for the zeroth harmonic $\rho(r)$ averaged over the phase ψ has the time-independent solution [7]

$$\rho_M(r)=\frac{1}{Z_M}\exp-\frac{1}{T}[H(r)+\sigma(r)], \quad (23)$$

where $H(r)=2\epsilon/\kappa^2(r)$,

$$\sigma=\pm 2\omega_r\sqrt{\epsilon M}\theta(1-\kappa)\left[\frac{1}{\kappa}-\Psi(\kappa)\right], \quad (24)$$

and

$$\rho_M(r)=\frac{1}{Z_M}\exp\left\{-\frac{1}{T}\left[\frac{2\epsilon}{\kappa^2}\pm 2\omega_r\sqrt{\epsilon M}\theta(1-\kappa)\times\left(\frac{1}{\kappa}-\Psi(\kappa)\right)\right]\right\}. \quad (25)$$

Here

$$\Psi(\kappa)=1-\int_\kappa^1\frac{du}{u^2}\left[\frac{\pi}{2E(u)}-1\right], \quad \Psi(0)=0.69. \quad (26)$$

The new solution can be expanded in azimuthal harmonics in $J=J_r+q$, ϕ variables

$$\rho_M(r)=\rho_0(J)+\sum_{m\neq 0} f_m(J,t)e^{-im\phi}. \quad (27)$$

The azimuthal harmonics $f_m(J,t)$, $J=J_r+q$ is

$$f_m(J,t)=\int\frac{d\phi' dJ'}{2\pi}\delta(J-J')e^{im\phi'}\rho(J',\phi',t), \quad (28)$$

where we introduce an additional integration $dJ'\delta(J-J')=dq\,\delta(q-q')$. The resonance solution $\rho(J,\phi)dJd\phi=drd\psi\rho_M(r)$ gives

$$f_m(J,t)=\int\frac{drd\psi}{2\pi}\delta[q-q(r,\psi)]\rho_M(r)e^{im\phi(r,\psi)}, \quad (29)$$

where $q(r,\psi)$ is defined in Eq. (20), and $\phi(r,\psi)=\omega_r t+\alpha(r,\psi)-\xi+\arg V_n/n$.

If there is only one resonance $\omega(J)=\Omega/n$ within the bunch rms size, averaging over fast oscillating terms leaves one nonzero harmonic, $m=n$:

$$f_n(q)=f_n(J_r+q,t)e^{-i\Omega t}=-\mathrm{sgn}(\omega_r')e^{i\arg V_n}\int\frac{drd\psi}{2\pi}e^{in\alpha(r,\psi)}\rho_M(r)\times\delta[q-q(r,\psi)]. \quad (30)$$

Replacing integration over $drd\psi$ by integration over $d\alpha d\kappa$,

$$drd\psi=\frac{4M\epsilon}{|q|\kappa^3}d\alpha d\kappa, \quad (31)$$

and introducing the new variable $p=\sqrt{4M\epsilon/\kappa^2}$, we get

$$f_n(q)=\mathrm{sgn}(\omega_r')e^{i\arg V_n/n}\Phi(q,\epsilon), \quad (32)$$

where

$$\Phi(q,\epsilon)=\int_{-\pi}^{\pi}d\beta\,\cos(2\beta)\int_0^\infty\frac{\sqrt{4M\epsilon}d\kappa}{2\pi\kappa^2\sqrt{1-\kappa^2\sin^2\beta}}, \quad (33)$$

$$\left\{\rho_M^+(\kappa)\delta\left[q-\sqrt{\frac{4M\epsilon}{\kappa^2}(1-\kappa^2\sin^2\beta)}\right]+\rho_M^-(\kappa)\delta\left[q+\sqrt{\frac{4M\epsilon}{\kappa^2}(1-\kappa^2\sin^2\beta)}\right]\right\}. \quad (34)$$

Here, $\beta=n\alpha/2$, $\rho_M(\kappa)=\rho_\pm$ at $\kappa<1$, and $\rho_M(\kappa)=\rho_s$ at $\kappa>1$,

$$\rho_\pm(p)=\frac{1}{Z_M}e^{-1/T[p^2/2M\pm\omega_r p\mp\omega_r p_s\Psi(p_s/p)]},$$

$$\rho_s(p)=\frac{1}{Z_M}e^{-p^2(q,\beta)/2MT}, \quad (35)$$

where $p=\sqrt{4\epsilon M}/\kappa$. The limits of integration in Eq. (34) for $q<0$ are given by the condition $q=J-J_r>-J_r$, and depend on $\kappa_{\min}=\sqrt{4M\epsilon/J_r^2}$ and $\kappa_0=\kappa_{\min}/\sqrt{1+\kappa_{\min}^2}$.

The function $\Phi(q,\epsilon)$ describes the spatial structure of the mode. It has a logarithmic singularity at the center of the separatrix and behaves as $\Phi\simeq 4\sqrt{M\epsilon}\rho_M(q)/q$ at large $q\gg\sqrt{\epsilon M}$.

The zeroth harmonic $\rho_0(J)$ in Eq. (27) can be similarly defined: $\rho_0(J)=\Phi_0(q,\epsilon)$, where Φ_0 is given by the expression Eq. (34) with the factor $\cos(2\beta)$ replaced by one.

Far away from the separatrix, where $\kappa\ll 1$, but $q\simeq\pm\sqrt{4M\epsilon}/\kappa$ is finite,

$$\rho_0(J)=\frac{1}{Z_M}e^{-1/T[H+\sigma]},$$

$$H(r)+\sigma(r)\simeq\frac{q^2}{2M}+\omega_r q\mp 2\omega_r\sqrt{\epsilon M}\Psi(0). \quad (36)$$

The resonance distribution Eq. (36) is different from the Haissinski solution: the number of particles increases at amplitudes larger than the resonance amplitude J_r, and decreases at the amplitudes $J<J_r$, compared with that of the Haissinski solution. There is a finite transition region with dimension $\Delta J\propto\sqrt{\epsilon M}$ that corresponds to a finite separtrix at $J=J_r$. Equation (36) written in variables x,p is not factorized, therefore, the rms energy spread $\langle p^2\rangle$ is different from the rms δ^2 of the Haissinski solution. The result looks like bunch heating, although the bunch is described by a steady-

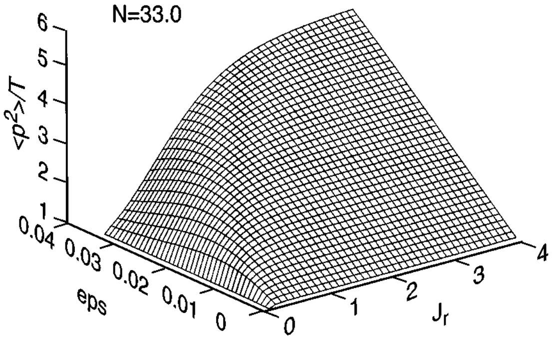

FIG. 1. RMS $\langle \delta^2 \rangle$ for the resonance solution.

state solution. Figure 1 illustrates the growth of $\langle p^2/T \rangle$ with ϵ. For the estimate, the average was calculated as the average of $\omega_0 J + \omega' J^2$ with normalized distribution functions $\rho_H = (1/Z_H)\exp[-H(J)/T]$ and $\rho_M = (1/Z_M)\exp\{-[H(J) \pm 2\omega_r \sqrt{\epsilon M}\Psi(0)]/T\}$. In the first case, $\langle p^2/T \rangle = 1$; the average calculated in the second case is shown in Fig. 1.

Equation (23) is derived for the case of a single resonance. It can be generalized for a case of multiple noninteractive resonances

$$\rho_0(J) = \frac{1}{Z_M}\exp - \frac{1}{T}\left[H(J) \mp \sum_{J_n < J} 2\omega_{r,n}\sqrt{\epsilon_n M}\Psi(0) \right], \tag{37}$$

where summation is over all resonances $J_r < J$ starting from $J = 0$.

It should be noted that for consistency, parameters ω_r and ω_r' should be defined for the distribution Eq. (36). In the present simulations these parameters are taken for the Haissinski distribution, which differs from Eq. (36) by a term of the order of $\sqrt{\epsilon M}$, which can be taken into account by iterations.

The new solution is the result of filamentation of azimuthal harmonics of the course-grain distribution function $\rho_M(r,\psi,t)$, starting from the initial condition $\rho_M(r,\psi,0) = \rho_H(J)$. The averaged mth harmonic

$$\langle \rho_M^m(t) \rangle \equiv \int dr \rho_M^m(r,t)$$

$$= \int \frac{d\psi dr}{2\pi} e^{im\psi - im\omega_M(r)t}\rho_H[J_r + q(r,\psi)] \tag{38}$$

decays in time as $\langle \rho_M^m(t) \rangle \propto \exp[-(m^2 T/2M)t^2]$.

Equations (27) and (32) show that the steady-state resonance solution $\rho(r)$ corresponds in the J,ϕ variables to a combination of a distorted Haissinski distribution, Eq. (36), plus a time-dependent resonance harmonic $f_n(J)e^{i(n\phi - \Omega t)}$, describing particles trapped in a separatrix. Other harmonics, equal on average to zero, may be important as well. The first harmonics, in particular, defines the time-dependence $\langle x \rangle$ of the bunch centroid,

$$\langle x \rangle = A\sin(\omega_r t + \arg f_1), \tag{39}$$

with amplitude $A = \int dJ f_1(J)\sqrt{2J/\omega_0}$.

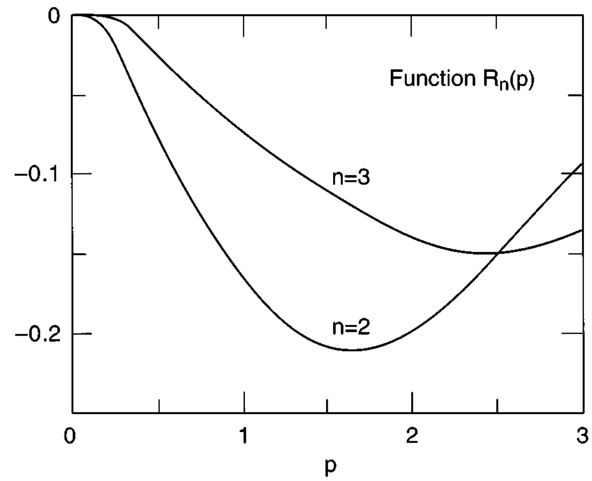

FIG. 2. $R_{nn}(J_r, J_r)$ as a function of J_r for $n=2$ and $n=3$.

The amplitude of the perturbation ϵ can now be determined from the condition of self-consistency. The self-consistent potential Eq. (2),

$$U_W = \lambda \int dx' dp' \rho(x',p') S[(x'-x)\sigma_0], \tag{40}$$

can be expanded in the azimuthal harmonics

$$U_W(x,t) = \sum V_m(J,t)\exp i(\Omega t - m\phi). \tag{41}$$

Equation (19) then defines ϵ in terms of the amplitude V_n,

$$\epsilon = -4\pi\lambda \int dq' \Phi(q',\epsilon) R_{nn}(J_r + q', J_r), \tag{42}$$

where

$$R_{lm}(J',J) = c_0 \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi i} \frac{Z(\zeta)}{\zeta} C_l^*(\zeta,J') C_m(\zeta,J) \tag{43}$$

and

$$C_m(\nu,J) = \int \frac{d\phi}{2\pi} e^{im\phi} e^{-i\nu\sigma_0 x(J,\phi)/c_0}. \tag{44}$$

Note that

$$C_m^*(\nu,J) = C_{-m}(-\nu,J),$$

$$R_{lm}^*(J',J) = R_{-l-m}(J',J) = R_{m,l}(J',J). \tag{45}$$

In a linear approximation, $R_{l,m}$ is given by Eq. (C6).

It is easy to see that, for $\epsilon \to 0$, $\Phi \propto \epsilon$ for $q > p_s$, and $\Phi \propto \sqrt{\epsilon}$ for $q < p_s$, $q/p_s \to$ const. Hence, the RHS in Eq. (42) is proportional to ϵ, and the nonzero solution exists only for sufficiently large λ. Figure 2 shows dependence of R_{nn} on J_r for two azimuthal modes, $m=2$ and $m=3$, and LEP broadband impedance (see below).

Which one of the two solutions, the Haissinski solution Eq. (16) or the resonance solution Eq. (25), is stable depends

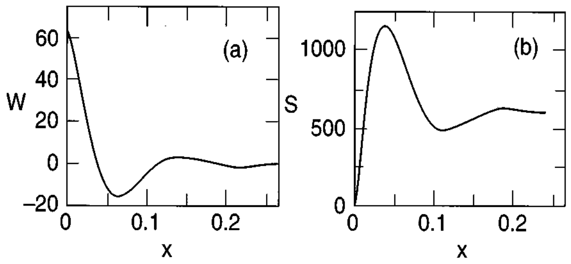

FIG. 3. Wake potential $W(z)$ and $S(z)$ for LEP.

on the minimum of the free energy $\Phi = E/T - S$, where E is the average energy per particle, $E = \langle H \rangle$, and S is the entropy,

$$S = -\langle \ln\rho \rangle = -\int dx dp \rho(x,p) \ln\rho(x,p). \quad (46)$$

The difference of the free energies depends on the parameters ϵ_f and on $\omega_r\sqrt{2M/T}$.

The free energy for the resonance solution is

$$\Phi_M = -\epsilon - \ln Z_M - \frac{1}{T}\langle \sigma \rangle, \quad (47)$$

where the angular brackets mean averaging with the resonance solution Eq. (25). Free-energy Eq. (47) should be compared with Φ_q, the free-energy for the Haissinski distribution calculated in the rotating frame

$$\Phi_q = \Phi_H - \left\langle \frac{\omega_r J}{T} \right\rangle, \quad (48)$$

or $\Phi_q = -\ln Z_q - \langle \omega_r q/T \rangle$ where brackets mean averaging over ρ_H given by Eq. (16). Equation (48) takes into account the transform given by Eq. (17).

The difference $\Delta\Phi = \Phi_q - \Phi_M$ was calculated numerically for LEP parameters: $E = 45$ GeV, $C = 26.66$ km, $\delta = 1.2\times10^{-3}$, $\alpha = 3.86\times10^{-4}$, $\sigma_0 = 2.0$ cm. The nominal bunch current 0.75 mA corresponds to $N_b = 4.2\times10^{11}$. The broadband impedance is described by the $Q=1$ model with $Z/n = 0.25\ \Omega$ and resonance frequency 2 GHz [3]. The wake potential and the function $S(x)$ of the Haissinski solution are shown in Fig. 3. Results of the Haissinski calculations for LEP are given in Table I.

Figures 4(a)–4(d) show results of numerical calculations for LEP with N_B in the range $N_B = (21.1 - 36.5)\times10^{10}$. The contour lines of a constant $\Delta\Phi$ are plotted in plane ϵ and J_r. They are superimposed with the lines where the RHS of Eq. (42) is equal to 0.9ϵ, 1.0ϵ, and 1.1ϵ. These lines are plotted for two azimuthal modes, $m=2$ and $m=3$. The maximum of $\Delta\Phi$ in the upper right corner of the plot can be approached only along these lines. Figure 4 shows that, at small N_B, the maximum reachable $\Delta\Phi$ corresponds to small amplitudes of the perturbation ϵ, and the distribution is, basically, the Haissinski solution. The situation is different when ω' approaches zero. In this case, large amplitudes of ϵ are possible and, more than that, Eq. (42) has two solutions for an azimuthal mode. At a given $\Delta\Phi$, these two solutions have different J_r, i.e., different frequencies $\omega_r = \omega(J_r)$. This difference is, however, small due to small ω'. The frequencies change while the amplitudes of perturbation ϵ grow adiabatically and they become closer to each other. It is reasonable to expect that, eventually, the separatrices of both solutions will overlap and destroy each other. The overlapping generates stochastic layers and increases the entropy of the system, which may change the temperature. The filamentation and synchrotron radiation damping bring the system back to the original state, with the Haissinski distribution function. The process then repeats itself displaying large amplitude nonlinear oscillations observed in experiments. As can be seen from Fig. 4, the overlapping of the separatrices of different azimuthal modes is possible, but is less important than the overlapping of different solutions of Eq. (42) for the same azimuthal mode. It is worthwhile to mention that Oide [12] came to a similar conclusion when solving numerically the linearized Vlasov equation. He also concluded that $\omega' = 0$ may be considered to be the indication of instability. This criterion can be expected because the stable (elliptic) and unstable (hyperbolic) points of a separatrix for the resonance Hamiltonian Eq. (18) in the plane α, q interchange when ω' changes sign.

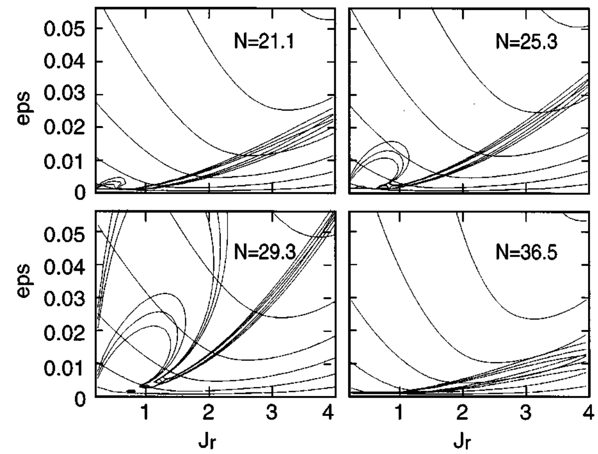

FIG. 4. Difference of free energies $\Delta\Phi$ and lines corresponding to the condition of self-consistency Eq. (42) for modes $n=2,3$ and different N_B.

TABLE I. Variation of the parameters of the Haissinski distribution with N_B.

$N_B\times10^{-10}$	V (MV)	ω_0	ω'	$\alpha^2 - 4\beta\omega^2$	$\alpha^2/\beta\omega^2$
21.1	35.0	1.04	0.356×10^{-02}	-0.0606	0.618
25.3	35.0	1.05	0.241×10^{-02}	-0.0727	0.741
29.3	35.0	1.06	0.895×10^{-03}	-0.0827	0.848
33.0	35.0	1.03	0.211×10^{-02}	-0.0459	0.693
36.5	35.0	1.12	-0.125×10^{-01}	-0.152	0.132

Figure 5 shows the contour plot of $\omega'(N,V)$ in the plane N_B, V. Crossing the line $\omega' = 0$ leads to instability. Area $\omega' < 0$ at large N_B does not necessarily mean bunch stability, but may correspond to another mechanism of instability related to appearance of the second minimum of the self-consistent potential.

Figure 6 shows dependence of ω', calculated for the Haissinski solution, on temperature T. ω' increases with T

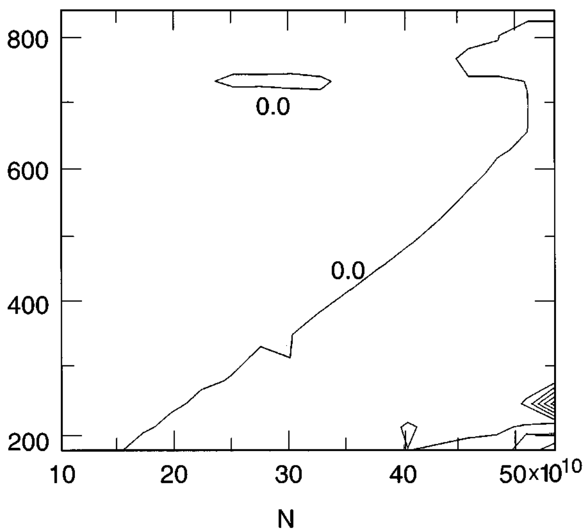

FIG. 5. Contour line $\omega'(N,V)=0$ in the plane V, N_B.

for $N_B > N_{th}$ and decreases for $N < N_{th}$, where $N_{th} \simeq 18.\times 10^{10}$ for LEP parameters. Decreasing ω' may lead to a thermal instability.

For the above speculation it is important that the system adiabatically follows variation of the parameters, in particular, the variation of ϵ. This will be true if the radiation damping is sufficiently large; otherwise, the rapid growth of ϵ may induce some oscillations around the resonance solution. This may explain the difficulty in obtaining sawtooth oscillations in numerical simulations.

V. BUNCH SPECTRUM

The experimentally observed bunch spectrum quite often has sidebands with unequal amplitudes that change in time in an irregular way. This result is quite unusual. The bunch spectrum $V(\omega)$ at the nth revolution harmonic has sidebands

$$\delta(\omega - n\omega_{rev} - \Omega)\int \rho(\Omega,z)e^{-inz/R}dz + \delta(\omega - n\omega_{rev} + \Omega)\int \rho^*(\Omega,z)e^{-inz/R}dz, \qquad (49)$$

defined by the harmonics of the bunch distribution

$$\rho(z,t) = \rho(\Omega,z)e^{-i\Omega t} + \rho^*(\Omega,z)e^{i\Omega t}. \qquad (50)$$

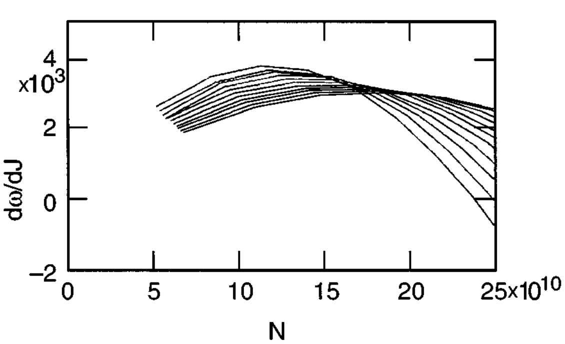

FIG. 6. Dependence of ω' on N_B for different temperatures T and fixed rf voltage $V = 350$ MV.

The amplitudes of the sidebands are equal if there is reversibility in time, $\rho(z,t) = \rho(z,-t)$. Unequal sidebands may be related to the mechanism of the instability.

Bunch spectrum measured by a BPM corresponds to the spectrum of a bunch current

$$V_{\mathrm{BPM}}(\omega) \propto \sum_{m=-\infty}^{\infty} \int dt e^{i(\omega - m\omega_{rev})t} S_m(t), \qquad (51)$$

where

$$S_m(t) = \int dx dp \rho(x,p,t) e^{-im(\omega_{rev}\sigma_0/c_0)x} = \int dJ d\phi \rho(J,\phi,t) e^{-im(\omega_{rev}\sigma_0/c_0)x(J,\phi)}. \qquad (52)$$

Substitute expansion Eq. (27) over azimuthal harmonics and use the definition of Eq. (44). The zeroth harmonic $\rho_0(J)$ gives the signal at the revolution harmonics. Harmonics ρ_m, with $m \neq 0$, give

$$S_m(t) = \sum_{k>0} \int dJ [f_k(J) e^{ik\omega_r t} C_{-k}(m\omega_{rev}, J) + f_k^*(J) e^{-i\omega_r t} C_k(m\omega_{rev}, J)]. \qquad (53)$$

The first term gives the signal at the revolution harmonics. The amplitudes of the upper and lower sidebands at the frequencies $\omega = m\omega_{rev} \pm (k/n)\Omega$ are proportional to

$$V_+ = \sum_{k>0} \int dJ f_k(J) C_k(m\omega_{rev}, J), V_- = \sum_{k>0} 2\pi \int dJ f_k^*(J) C_{-k}(m\omega_{rev}, J), \qquad (54)$$

where

$$f_k(J) = \int \frac{dr d\psi}{2\pi} \delta[J - J_r - q(r,\psi)] \rho_M(r) e^{-ik\alpha(r,\psi)}. \qquad (55)$$

Notice that $C_{-k} = (-1)^k C_k$; see Appendix C. If there is only one azimuthal harmonic $f_n(J)$, as in Eq. (30), $f_k = f_k^*$ and the amplitudes $V_\pm$ are equal, providing the difference in the impedance $Z(\omega)$ at the sideband frequencies is negligible. The situation is different if there are two solutions of Eq. (42). In this case, the nth azimuthal harmonics corresponds to two radial modes

$$f_n = [(-)^n f_1 e^{i\Omega_1 t} + f_1^* e^{-i\Omega_1 t}] C_n(m\omega_r, J_1) + (1 \to 2), \qquad (56)$$

with separatrices located at the amplitudes $J_{1,2}$. For small ω', the frequencies of two modes $\Omega_{1,2} = \omega(J_{1,2})$ cannot be resolved experimentally. The BPM signal would have time-dependent beating with frequency $\Delta\Omega = \Omega_1 - \Omega_2$ and unequal amplitudes

$$V_+ = f_1 C_1^* \left[1 + \frac{f_{n2}}{f_{n1}} \left(\frac{C_2}{C_1} \right)^* e^{-i\Delta\Omega t} \right];$$

$$V_{-}=f_1C_1\left[1+\frac{f_{n2}}{f_{n1}}\left(\frac{C_2}{C_1}\right)e^{-i\Delta\Omega t}\right], \tag{57}$$

because the coefficients C_n are complex due to the nonlinearity of the self-consistent potential; see Eq. (C5).

Another possible explanation of the unequal sideband amplitude may be related to the excitation of the motion of the bunch centroid for the resonance solution.

VI. DISCUSSION

The general statement for a system in thermodynamic equilibrium is [11] ''No internal macroscopic motion is possible in a state of equilibrium.'' However, the system under consideration is specific. First, the N-particle system does not have a Hamiltonian because the third Newton's law is not applicable for the interaction with the wake potential. A particle in a self-consistent potential can, nevertheless, be described by the Fokker-Plank equation [7]. The resonance solution appears as a result of spontaneously breaking symmetry. This solution describes a self-consistent regime when particles trapped in a separatrix of an azimuthal mode with eigenfrequency Ω produce a perturbation of the self-consistent potential equivalent to the excitation of such a mode. In the rotating frame, the resonance solution is a steady-state solution. In the original J,ϕ variables, this corresponds to a certain combination of a modified zeroth and nth azimuthal harmonics if there are resonance particles with frequencies $\omega(J)\simeq\Omega/n$.

At large bunch current the resonance solution may have lower free energy than the Haissinski solution. The minimum of free energy and the condition of self-consistency define the parameters of the resonance solution ϵ and Ω. A bunch in the new state would have an rms energy spread different from the rms energy spread of the Haissinski solution. The process of transformation from one solution to the other can be adiabatic, provided the damping time is large. A simple nonlinear dynamics exists only for a single resonance. It is easy to speculate that, depending on the potential well distortion and parameters of the bunch, several resonances can be excited simultaneously. The interaction between resonances at large amplitudes may lead to overlapping of the separatrices, and their destruction. In this case, trapped particles become free, filament, and produce a dynamic heating of the bunch, which must be distinguished from the quasistatic change of the rms energy spread of a single mode.

ACKNOWLEDGMENTS

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APPENDIX A: SOME RELATIONS FOR A NONLINEAR PENDULUM

The following relations are useful:

For $0<\kappa<1$ (unbounded motion):

$$H=\frac{2\epsilon}{\kappa^2},\quad \omega_M=\frac{dH}{dr}=\frac{\pi}{\kappa K(\kappa)}\sqrt{\frac{\epsilon}{M}}, \tag{A1}$$

$$r=\frac{4}{\pi}\sqrt{\epsilon M}\frac{E(\kappa)}{\kappa},\quad \frac{dr}{d\kappa}=-\frac{4}{\pi}\sqrt{\epsilon M}\frac{K(\kappa)}{\kappa^2}, \tag{A2}$$

$$\psi=\pm\frac{\pi}{nK(\kappa)}F\left[\frac{n\alpha}{2},\kappa\right],\quad \frac{\partial\psi}{\partial\alpha}=\frac{\pi}{qK(\kappa)}\sqrt{\frac{\epsilon M}{\kappa^2}}, \tag{A3}$$

where q is defined by Eq. (20).

For $\kappa>1$ (bounded motion):

$$r=\frac{4}{\pi}\sqrt{\epsilon M}[E(1/\kappa)-(1-1/\kappa^2)K(1/\kappa)],$$

$$\frac{dr}{d\kappa}=-\frac{4}{\pi}\sqrt{\epsilon M}\frac{K(1/\kappa)}{\kappa^3}, \tag{A4}$$

$$\psi=\pm\frac{\pi}{nK(1/\kappa)}F\left[\arcsin\left(\kappa\sin\frac{n\alpha}{2}\right),\frac{1}{\kappa}\right],\quad \frac{\partial\psi}{\partial\alpha}=\frac{\pi\sqrt{\epsilon M}}{qK(1/\kappa)}. \tag{A5}$$

Elliptic integrals K,E and elliptic functions F,E are defined in Gradshteyn [10].

APPENDIX B: TRANSFORMATION OF THE FOKKER-PLANK EQUATION

First, we follow Shonfeld [6] to transform the Fokker-Plank equation, Eq. (4), to the variables J,ϕ.

The following identities are valid for an arbitrary function $F(x,p,t)$:

$$\frac{\partial F}{\partial p}=\{x,F\}_{x,p}=\{x,F\}_{\phi,J}=\frac{\partial}{\partial J}\left(\frac{\partial x}{\partial\phi}F\right)-\frac{\partial}{\partial\phi}\left(\frac{\partial x}{\partial J}F\right), \tag{B1}$$

so that, for the phase average values,

$$\left\langle\frac{\partial F}{\partial p}\right\rangle=\frac{\partial}{\partial J}\left\langle\frac{\partial x}{\partial\phi}F\right\rangle. \tag{B2}$$

In particular, the RHS of Eq. (4) is

$$X_{\mathrm{RHS}}=\left\langle\left[D\frac{\partial^2\rho}{\partial p^2}+\gamma_d\frac{\partial p\rho}{\partial p}\right]\right\rangle=\frac{\partial}{\partial J}\left\langle\frac{\partial x}{\partial\phi}\left[D\frac{\partial\rho}{\partial p}+\gamma_d p\rho\right]\right\rangle. \tag{B3}$$

Substitute here $\partial\rho/\partial p=\{x,\rho\}_{(x,p)}=\{x,\rho\}_{(\phi,J)}$. For $\rho=\rho(J,t)$,

$$X_{\mathrm{RHS}}=\frac{\partial}{\partial J}\left[D\left\langle\left(\frac{\partial x}{\partial\phi}\right)^2\right\rangle\frac{\partial\rho}{\partial J}+\gamma_d\left\langle p\frac{\partial x}{\partial\phi}\right\rangle\rho(J)\right]. \tag{B4}$$

The identity $\partial x/\partial\phi=\{J,x\}=\partial J/\partial p$ gives

$$p\frac{\partial x}{\partial\phi}=p\frac{\partial J}{\partial p}=\frac{\partial}{\partial p}(pJ)-J. \tag{B5}$$

Equations (B2) and (B5) give, for the phase average,

$$\left\langle p\,\frac{\partial x}{\partial \phi}\right\rangle = \frac{\partial}{\partial J}\left\langle \frac{\partial x}{\partial \phi} pJ\right\rangle - J = J\frac{\partial}{\partial J}\left\langle p\,\frac{\partial x}{\partial \phi}\right\rangle + \left\langle p\,\frac{\partial x}{\partial \phi}\right\rangle - J. \tag{B6}$$

Comparison of the left- and right-hand sides defines the second term in Eq. (B4):

$$\left\langle p\,\frac{\partial x}{\partial \phi}\right\rangle = J. \tag{B7}$$

The momentum p can be written

$$p = \frac{\partial H}{\partial p} = \{x,H\} = \omega(J)\,\frac{\partial x}{\partial \phi}, \tag{B8}$$

provided the Hamiltonian is

$$H(x,p) = \frac{p^2}{2} + U(x) = H(J), \quad \omega(J) = \frac{\partial H}{\partial J}. \tag{B9}$$

Equations (B7) and (B8) define the first term in Eq. (B4):

$$\left\langle p\,\frac{\partial x}{\partial \phi}\right\rangle = \omega(J)\left\langle \left(\frac{\partial x}{\partial \phi}\right)^2\right\rangle = J. \tag{B10}$$

The Fokker-Plank equation takes the form

$$\frac{\partial \rho}{\partial t} + \{H,\rho\} = \frac{\partial}{\partial J}\left[D\frac{J}{\omega(J)}\,\frac{\partial \rho}{\partial J} + \gamma_d J\rho(J)\right]. \tag{B11}$$

Consider now the Hamiltonian Eq. (15). The RHS of the Fokker-Plank equation for $\rho = \rho(r)$ can be written in r,ψ variables similar to Eq. (B4):

$$X_{\mathrm{RHS}} = \frac{\partial}{\partial r}\left[D\left\langle \left(\frac{\partial x}{\partial \psi}\right)^2\right\rangle \frac{\partial \rho}{\partial r} + \gamma_d\left\langle p\,\frac{\partial x}{\partial \psi}\right\rangle \rho\right]. \tag{B12}$$

Similar to Eq. (B7), the identities $\partial x/\partial\psi = \{r,x\} = \partial r/\partial p$ and

$$\begin{aligned} p\frac{\partial x}{\partial \psi} &= p\,\frac{\partial r}{\partial p} = -r + \frac{\partial}{\partial p}(pr) \\ &= -r + \frac{\partial}{\partial r}\left(\frac{\partial x}{\partial \psi}pr\right) - \frac{\partial}{\partial \psi}\left(\frac{\partial x}{\partial r}pr\right) \end{aligned} \tag{B13}$$

give, after averaging over ψ,

$$\left\langle p\,\frac{\partial x}{\partial \psi}\right\rangle = -r + \frac{\partial}{\partial r}\left[\left\langle p\,\frac{\partial x}{\partial \psi}\right\rangle r\right]. \tag{B14}$$

Hence,

$$\left\langle p\,\frac{\partial x}{\partial \psi}\right\rangle = r. \tag{B15}$$

The Fokker-Plank equation takes the form

$$\frac{\partial \rho}{\partial t} + \{H,\rho\}_{r,\psi} = \gamma_d\,\frac{\partial}{\partial r}\left[T\left\langle \left(\frac{\partial x}{\partial \psi}\right)^2\right\rangle + r\rho\right]. \tag{B16}$$

The coefficient $d \equiv \langle(\partial x/\partial\psi)^2\rangle$ can be related to the average $v \equiv \langle(\partial x/\partial\psi)(\partial x/\partial\phi)\rangle$. Consider the Hamiltonian

$$\begin{aligned} H(x,p,t) &= H(J,\phi,t) = H(q,\alpha) + \omega_r(J_r+q) \\ &= H(r) + \omega_r(J_r+q), \end{aligned} \tag{B17}$$

where $H(J,\phi,t)$ and $H(q,\alpha)$ are defined in Eqs. (15) and (18), correspondingly. The momentum

$$p = \{x,H\}_{x,p} = \{x,H+\omega_r q\}_{\psi,r} = \omega_M\,\frac{\partial x}{\partial \psi} + \omega_r\{x,q\}, \tag{B18}$$

or $p = \omega_M\partial x/\partial\psi + \omega_M\partial x/\partial\phi$, where $\omega_M = \partial H/\partial r$. Hence,

$$r = \left\langle p\,\frac{\partial x}{\partial \psi}\right\rangle = \omega_M d + \omega_r v. \tag{B19}$$

The solution of Eq. (B16) for the zeroth azimuthal harmonic $\rho(r)$ in a steady-state is

$$\rho_M(r) = \frac{1}{Z_M}e^{-1/T[H(r)+\sigma_M(r)]}, \tag{B20}$$

where σ_M is defined by Eqs. (B16) and (B19) [7],

$$\frac{d\sigma_M}{dr} = \omega_r\,\frac{v}{d}. \tag{B21}$$

The coefficients v and d can be found from the canonical transform $x,p\to\phi,J$, which defines the coefficients of expansion

$$x(J,\phi) = \sum_m a_m(J)e^{im\phi}. \tag{B22}$$

Then,

$$\frac{\partial x}{\partial \phi} = \sum im a_m e^{im\phi}, \quad \frac{\partial x}{\partial \psi} = \sum_m \left[im\,\frac{\partial \alpha}{\partial \psi} + \frac{\partial q}{\partial \psi}\,\frac{da_m}{dJ}\right] e^{im\phi}, \tag{B23}$$

where α and q are defined in Eq. (17). Averaging over fast oscillations gives

$$\left\langle \frac{\partial x}{\partial \phi}\,\frac{\partial x}{\partial \psi}\right\rangle = \sum \left[m^2|a_m|^2\left\langle \frac{\partial \alpha}{\partial \psi}\right\rangle - ima_m^* a_m'\left\langle \frac{\partial q}{\partial \psi}\right\rangle\right]. \tag{B24}$$

The last term here is equal to zero because $q(r,\psi)$ is periodic with ψ. For the same reason, $\langle\partial x/\partial\psi\rangle = 0$ in the separatrix, and is equal to ± 1 for $q\pm 0$ outside of the separatrix. Hence,

$$\left\langle \frac{\partial x}{\partial \phi}\,\frac{\partial x}{\partial \psi}\right\rangle = \pm\,\theta(1-\kappa)\sum m^2|a_m|^2, \quad q\pm 0. \tag{B25}$$

The average

$$\left\langle \left(\frac{\partial x}{\partial \psi}\right)^2\right\rangle = \sum \left[m^2|a_m|^2\left\langle \left(\frac{\partial \alpha}{\partial \psi}\right)^2\right\rangle + \left\langle \left(\frac{\partial q}{\partial \psi}\right)^2\right\rangle \left| a_m'\right|^2\right], \tag{B26}$$

because the cross-terms $\langle(\partial\alpha/\partial\psi)(\partial q/\partial\psi)\rangle = 0$. The first term

$$\left\langle \left(\frac{\partial \alpha}{\partial \psi} \right)^2 \right\rangle = \frac{4}{\pi^2} K(\kappa) E(\kappa), \quad \kappa < 1. \tag{B27}$$

The average

$$\left\langle \left(\frac{\partial q}{\partial \psi} \right)^2 \right\rangle = \epsilon M \frac{M^2}{\pi^2} \langle \sin^2(n\alpha) \rangle_\psi , \tag{B28}$$

and is small for small ϵ. Neglecting this term, we get

$$\frac{v}{d} = \pm \theta(1-\kappa) \frac{\pi^2}{4K(\kappa)E(\kappa)}. \tag{B29}$$

Equation (B21) then gives Eq. (24),

$$\sigma_M = \pm \theta(1-\kappa)\ 2\omega_r \sqrt{\epsilon M} \left[\frac{1}{\kappa} - \Psi(\kappa) \right]. \tag{B30}$$

APPENDIX C: DERIVATION OF R_{nn}

Coefficients $C_n(\nu,J)$ in Eq. (44) are given in terms of a trajectory $x(J,\phi)$ of the Hamiltonian Eq. (1), $H(J)=H(x,p)=p^2/2+U(x)$. Approximate the self-consistent potential by a polynomial

$$U(x) = U_{\min} + \frac{\omega^2 (x-x_{\min})^2}{2} + \frac{\alpha}{3}(x-x_{\min})^3 + \frac{\beta}{4}(x-x_{\min})^4, \tag{C1}$$

with parameters $U_{\min}$, $x_{\min}$, α, and β depending on N_b. For small nonlinearities, the trajectory can be written in series over $A=\sqrt{2J/\omega}$:

$$x(J,\phi) = A\left\{ \sin\phi - \frac{\alpha A}{2\omega^2} - \frac{\alpha A}{6\omega^2}\cos 2\phi - \frac{A^2}{16\omega^2}\left(\frac{\alpha^2}{3\omega^2} - \frac{\beta}{2} \right) \sin 3\phi + \cdots \right\}, \tag{C2}$$

$$\frac{d\phi}{dt} = \omega + \left(\frac{3\beta}{2} - \frac{5\alpha^2}{3\omega^2} \right) \frac{A^2}{4\omega}. \tag{C3}$$

In this approximation, $C_m(\nu,J)$ is

$$C_m(\nu,J) = J_m(\nu a)\left[1 + \frac{i\nu\sigma_0\alpha}{2\omega^2} A^2 \right] + \frac{i\nu\sigma_0\alpha}{12\omega^2} A^2 [J_{m-2}(\nu a) + J_{m+2}(\nu a)] \tag{C4}$$

$$+ \frac{\nu\sigma_0}{32\omega^2}\left(\frac{\beta}{2} - \frac{\alpha^2}{3\omega^2} \right) A^3 [J_{m-3}(\nu a) - J_{m+3}(\nu a)], \tag{C5}$$

where $a=\sigma_0 A/c_0$, and J_m is the Bessel function. Note that $C_m^*(-\nu,J)=(-1)^m C_m(\nu,J)$, $C_{-m}(\omega)=(-1)^m C_m(\omega)$, and $R_{-l,-m}(J',J)=(-)^{m+l} R_{lm}(J',J)$. Hence, $R_{nn}^* = R_{nn}$, and the RHS of Eq. (42) for ϵ is real.

In the linear approximation,

$$R_{nn}(J,J) = c_0 \int_0^\infty \frac{d\nu}{\pi} \frac{\mathrm{Im} Z(\nu)}{\nu} J_n^2(\nu a). \tag{C6}$$

For broadband $Q=1$ model of the impedance

$$Z(\omega) = -i\xi \left[\frac{1}{\omega - \omega_n - i\gamma_n} + \frac{1}{\omega + \omega_n - i\gamma_n} \right], \tag{C7}$$

with $\gamma/\omega_n = 1/2$,

$$R_{nn}(J,J) = \frac{2}{Z_0}\left(\frac{Z}{n} \right) \frac{\omega_n}{\omega_{\text{rev}}} \int_0^\infty \frac{dx}{x} J_n^2\left(\frac{\omega_n a}{c_0} x \right) \times \left[\frac{1-x}{(x-1)^2 + 1/4} - \frac{1+x}{(1+x)^2 + 1/4} \right], \tag{C8}$$

where Z/n defines the inductive low-frequency behavior of the impedance at $\omega \ll \omega_n$, $Z/n = (2i\xi\omega_{rev}/\omega_n^2)\nu$.

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